GAUSS and Recovery of Interblock, Intercolumn, and Intergradient Information

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ABSTRACT

Using a GAUSS software package, version 1.49B, it is shown how to obtain the sums of squares for an analysis of variance table, how to obtain the variance-covariance matrix for the adjusted effects, and how to obtain an average variance of a difference between two effects. This is done for an incomplete block experiment design with v treatments in b incomplete blocks of size k in r complete replicates, for a lattice square or rectangle experiment design, and for an incomplete block design with linear trends within the incomplete blocks. Interblock, interrow and column, and interblock and intergradient information are recovered.

1. Introduction

To illustrate the programming in GAUSS to obtain statistical analyses, we use the following three examples:

(i) An incomplete block design with $v = 9$ treatments in $b = 3$ incomplete blocks of size $k = 3$ in $r = 3$ replicates (complete blocks),

(ii) A lattice square experiment design with $v = 16$ treatments in $k = 4$ rows and $c = 4$ columns in $r = 5$ replicates, and

(iii) An alternative analysis for (ii) considering rows and linear trends within each of the rows using the data in (ii).

Note that the programs are general in that the values for v, b, k, r, and c can be substituted in the programs. Likewise, the design matrix X and the various incidence matrices will be determined by the particular design used.

Key words and phrases: Incomplete block design; Lattice square design; Lattice rectangle design; Gradients in blocks; Average variance; ANOVA.

2. GAUSS Program For an Incomplete Block Experiment Design

A linear model for an incomplete block experiment design (IBED) is:

$$Y_{ghi} = \mu + \rho_g + \beta_{gh} + \tau_i + \epsilon_{ghi} , \tag{2.1}$$

where Y_{ghi} is the response for treatment i in incomplete block gh, μ is a general mean effect, ρ_g is the gth replicate effect, $g = 1,\cdots,r$, β_{gh} is the hth incomplete block effect in replicate g, $h = 1,\cdots,b$, τ_i is the effect of the ith treatment, $i = 1,\cdots,v$, and ϵ_{ghi} is a random error effect distributed with mean zero and variance σ^2_ϵ.

If $\mathbf{X}[rv, (r+rb+v+1) = \mathbf{X}[27, 24]$ is the design matrix and $\mathbf{Y}[rv, 1] = \mathbf{Y}[27, 1]$ is a column vector of the observations or responses, then the various totals are obtained as $\mathbf{X'Y}$. The solution for effects, $\boldsymbol{\alpha}$, is given by

$$\hat{\boldsymbol{\alpha}} = (\mathbf{X'X} - \mathbf{J0})^{-1}\mathbf{X'Y}) , \tag{2.2}$$

where

$$\mathbf{J0} = \left[\begin{array}{c|ccc} \mathbf{0}_{rv\times 1} & \mathbf{0}'_{1\times(rv-1)} & & \\ \hline & \mathbf{0}_{r\times r} & \mathbf{I}_r * \mathbf{J}_{1\times b} & \mathbf{J}_{r\times v} \\ & \mathbf{J}_{b\times 1} * \mathbf{I}_r & \mathbf{0}_{rb\times rb} & \mathbf{J}_{rb\times v} \\ & \mathbf{J}_{v\times r} & \mathbf{J}_{v\times rb} & \mathbf{0}_{v\times v} \end{array}\right] . \tag{2.3}$$

J is a matrix of ones, **I** is the identity matrix, **0** is a matrix of zeros, and $*$ denotes the Kronecker product. Equation (2.3) makes use of the following:

$$\sum_i \hat{\tau}_i = \sum_g \hat{\rho}_g = \sum_h \hat{\beta}_{gh} = 0 . \tag{2.4}$$

When there are no missing values

$$\hat{\mu} + \hat{\rho}_g = \bar{y}_{g\cdot\cdot} \quad \text{and} \quad \hat{\mu} = \bar{y}_{\cdot\cdot\cdot} . \tag{2.5}$$

Otherwise, $\hat{\mu} + \hat{\rho}_g$ is obtained from (2.2).

To simplify the presentation and program, we use $Y_{ghi} - \bar{y}_{g\cdot\cdot}$ or $Y_{ghi} - \hat{\mu} - \hat{\rho}_g$ values to obtain incomplete block and treatment totals, solutions, and sums of squares. Using the values minus the replicate mean, we have

$$\begin{bmatrix} \mathbf{kI}_{rb} & \mathbf{NB}_{rb \times v} \\ \mathbf{NB}'_{v \times rb} & \mathbf{rI}_v \end{bmatrix} \begin{bmatrix} \mathbf{BeT} \\ \mathbf{TeB} \end{bmatrix} = \begin{bmatrix} \mathbf{YB} \\ \mathbf{YT} \end{bmatrix}, \tag{2.6}$$

where **BeT** is the vector of incomplete block effects eliminating treatment effects, **TeB** is the vector of treatment effects eliminating incomplete block effects (intrablock treatment effects), **NB** is the block by treatment incidence matrix, **YB** is the vector of incomplete block totals for $Y_{ghi} - \bar{y}_{g\cdot\cdot}$ values, and **YT** is the vector of treatment totals for $Y_{ghi} - \bar{y}_{g\cdot\cdot}$ values. Replacing **NB** with $\mathbf{NB} - \mathbf{J}_{rb \times v}$, a solution for **BeT** and **TeB** effects is now possible. The block eliminating treatment sum of squares, BsseT, is obtained as

$$\mathrm{BsseT} = \left(\mathbf{YB} - \mathbf{NB} \times \mathbf{Yt}/\mathrm{r}\right)' \left[\mathbf{kI}_{rb} - \mathbf{NB} \times \mathbf{NB}'/\mathrm{r} + \mathbf{J}_{rb \times rb}/\mathrm{r}\right]^{-1} \times \left(\mathbf{YB} - \mathbf{NB} \times \mathbf{YT}/\mathrm{r}\right). \tag{2.7}$$

The block ignoring treatment sum of squares, BssiT, is obtained as $\mathbf{YB}' \times \mathbf{YB}/\mathrm{k}$. The treatment eliminating block sum of squares, TsseB, is

$$\mathrm{TsseB} = \left(\mathbf{YT} - \mathbf{NB}' \times \mathbf{YB}/\mathrm{k}\right)' \left[\mathbf{rI}_{v} - \mathbf{NB}' \times \mathbf{NB}/\mathrm{k} - \mathbf{J}_{v \times v}/\mathrm{k}\right]^{-1} \times \left(\mathbf{YT} - \mathbf{NB}' \times \mathbf{YB}/\mathrm{k}\right). \tag{2.8}$$

The treatment ignoring block sums of squares, TssiB, is $\mathbf{YT}' \times \mathbf{YT}/\mathrm{r}$. The sums of squares obtained from the GAUSS program are given in Table 1 for the following design and data.

	Replicate 1			Replicate 2			Replicate 3		
Block	11	12	13	21	22	23	31	32	33
	(1) 6	(4) 12	(7) 12	(1) 9	(2) 7	(3) 11	(1) 9	(2) 10	(3) 8
	(2) 8	(5) 10	(8) 12	(4) 9	(5) 11	(6) 13	(5) 11	(6) 12	(4) 10
	(3) 7	(6) 11	(9) 12	(7) 9	(8) 9	(9) 12	(9) 10	(7) 11	(8) 9

Treatment members are in parentheses. The Y_{ghi} values were constructed from the following effects:

	BeT	TeB	
$\hat{\mu} = 10$	$\beta_{11} = -2$	$\tau_1 = -1$	$\epsilon_{111} = -1$
$\hat{\rho}_1 = 0$	$\beta_{12} = 0$	$\tau_2 = -1$	$\epsilon_{112} = 1$
$\hat{\rho}_2 = 0$	$\beta_{13} = 2$	$\tau_3 = -1$	$\epsilon_{124} = 1$
$\hat{\rho}_3 = 0$	$\beta_{21} = -1$	$\tau_4 = 1$	$\epsilon_{125} = -1$
	$\beta_{22} = -1$	$\tau_5 = 1$	$\epsilon_{211} = 1$
	$\beta_{23} = 2$	$\tau_6 = 1$	$\epsilon_{214} = 1$
	$\beta_{31} = 0$	$\tau_7 = 0$	$\epsilon_{222} = -1$
	$\beta_{32} = 1$	$\tau_8 = 0$	$\epsilon_{225} = 1$
	$\beta_{33} = -1$	$\tau_9 = 0$	

All 19 remaining ϵ_{ghi} values are zero. Hence the intrablock sum of squares is 8 as given in Table 1. The **BeT** values obtained in the GAUSS program in Table A1 are those given above and likewise for the **TeB** values. (Note: In order to check programs, use of a constructed example with known values is quite helpful.)

The expected value for the block eliminating treatment mean square is

$$\sigma_\epsilon^2 + k(r-1)\sigma_\beta^2/r \, . \tag{2.9}$$

For the above data set, $E = \hat{\sigma}_\epsilon^2 = 0.8$ and $B = (16/3 - 0.8) \big/ 3(3-1)/3 = 2.2667$. To recover interblock information, we simply replace k in (2.6) by k + E/B and proceed with the solution. Thus

$$\mathbf{TeBr} = \left[r\mathbf{I}_v - \mathbf{NB}' \times \mathbf{NB} \big/ (k + E/B) + \mathbf{J}_{v \times v}/(k+E/B) \right]^{-1} \left[\mathbf{YT} - \mathbf{NB}' \times \mathbf{YB} \big/ (k + E/B) \right] . \tag{2.10}$$

The variance-covariance matrix for treatment effects recovering interblock information is

$$\mathbf{varcov} = E\left[r\mathbf{I}_v - \mathbf{NB}' \times \mathbf{NB} \big/ (k + E/B) + \mathbf{J}_{v \times v}/(k+E/B) \right]^{-1} . \tag{2.11}$$

The 1/v th root of the determinant of (2.11) times two is less than or equal to the average variance of a difference between two treatment effects recovering interblock information. The correct average variance of difference (see Federer, 1955, p. 313) is 0.70 versus 0.68 obtained as an approximation. If all diagonal elements are equal and if all off-diagonal elements are zero in (2.11), then the correct average variance of a difference is obtained.

Table 1. Analysis of variance (ANOVA) for an incomplete block design.

Source of variation	d.f.[1]	Sum of squares	Mean square
Total	rv = 27	Y′Y = 2786	—
Correction for mean	1	$270^2/27 = 2700$	—
Replicate	(r–1) = 2	0	—
Treatment ignoring block = TssiB	v–1 = 8	46	—
Block eliminating treatment = BsseT	r(b–1) = 6	32	16/3
Intrablock error	(r–1)(v–1) –r(b–1) = 10	8	0.8
Block ignoring treatment = BssiT	6	66	—
Treatment eliminating block = TsseB	8	12	1.5

[1] d.f. = degrees of freedom

$\hat{\sigma}_\epsilon^2 = \mathrm{E} = 0.8$

$\hat{\sigma}_\beta^2 = \mathrm{B} = (16/3 - 0.8) \Big/ k(r-1)/r = 2.667$

$\mathrm{E/B} = 0.8/2.667 = 0.300$

3. GAUSS Program For a Lattice Square or Lattice Rectangle Experiment Design

For this design we use the following linear response model,

$$Y_{ghij} = \mu + \rho_g + \beta_{gh} + \gamma_{gi} + \tau_j + \epsilon_{ghij} \ , \tag{3.1}$$

where β_{gh} is the kth row in the gth replicate effect, γ_{gi} is the ith column in the gth replicate effect, and the remaining effects are as for (2.1). A GAUSS program for obtaining an analysis of variance table, for obtaining treatment effects recovering row and column information, and for obtaining the variance-covariance matrix, is given in Table A2.

The data set used to illustrate the program was obtained from Table 12.5 of Cochran and Cox (1957). Using the $Y_{ghij} - \bar{y}_{g\cdots}$ values as in the previous section, the row-column-treatment matrix after applying the restrictions that sums of effects are zero, becomes:

$$\begin{bmatrix} c\mathbf{I}_{rk} & \mathbf{0}_{rk\times rc} & \mathbf{RT}-\mathbf{J}_{rk\times v} \\ \mathbf{0}_{rc\times rk} & k\mathbf{I}_{rc} & \mathbf{CT}-\mathbf{J}_{rc\times v} \\ \mathbf{RT}' & \mathbf{CT}' & r\mathbf{I}_v \end{bmatrix} \begin{bmatrix} \mathbf{ReTC} \\ \mathbf{CeTR} \\ \mathbf{TeRC} \end{bmatrix} = \begin{bmatrix} \mathbf{YR} \\ \mathbf{YC} \\ \mathbf{YT} \end{bmatrix} . \tag{3.2}$$

The above results in intrarow-column treatment effects, **TeRC**. The various sums of squares obtained from the GAUSS program from the data in Table 2 are summarized in the ANOVA in Table 3. (Note: The total sum of squares was obtained using a pocket calculator.) The expected value of the row eliminating treatment and column mean squares is

$$\sigma_\epsilon^2 + c(r-2)\,\sigma_\beta^2/(r-1) \tag{3.3}$$

and of the column eliminating row and treatment mean square is

$$\sigma_\epsilon^2 + k(r-2)\,\sigma_\gamma^2/(r-1) \ . \tag{3.4}$$

Then, $\hat{\sigma}_\beta^2 = R$ and $\hat{\sigma}_\gamma^2 = C$ are estimated as

$$R = (68.450-22.672) \big/ 4(3)/4 = 15.259$$

and

$$C = (37.306-22.672) \big/ 4(3)/4 = 4.878 \ .$$

Then

$$c + E/R = 4 + 22.672/15.259 = 5.485$$

and

$$k + E/C = 4 + 22.672/4.878 = 8.648 \ .$$

Table 2. Percentage of Squares Attacked by Boll Weevils for a 4 × 4 Lattice Square in (k + 1) Replications

Boll weevil infestation

									Row totals = $Y_{gh\cdot\cdot}$	$Y_{gh\cdot\cdot}-4\bar{y}_{g\cdots}$
Square I										
	(10)	9.0	(12)	20.3	(9)	17.7	(11)	26.3	73.3	32.475
	(2)	4.7	(4)	9.0	(1)	7.3	(3)	8.3	29.3	−11.525
	(14)	9.0	(16)	6.7	(13)	11.7	(15)	4.3	31.7	− 9.125
	(6)	4.0	(8)	5.0	(5)	5.7	(7)	14.3	29.0	−11.825
$Y_{g\cdot i\cdot}$ = Column totals		26.7		41.0		42.4		53.2	163.3	0
$Y_{1\cdot i\cdot}-4\bar{y}_{1\cdots}$		−14.125		0.175		1.575		12.375	0	
Square II										
	(5)	19.0	(12)	8.7	(15)	13.0	(2)	15.7	56.4	15.375
	(10)	12.0	(7)	6.0	(4)	15.3	(13)	12.0	45.3	4.275
	(16)	12.7	(1)	6.3	(6)	1.7	(11)	13.0	33.7	− 7.325
	(3)	3.7	(14)	3.7	(9)	8.0	(8)	13.3	28.7	−12.325
Column totals		47.4		24.7		38.0		54.0	164.1	0
$Y_{2\cdot i\cdot}-4\bar{y}_{2\cdots}$		6.375		−16.325		−3.025		12.975	0	
Square III										
	(10)	17.0	(15)	7.0	(8)	10.3	(1)	1.3	35.6	− 9.925
	(9)	11.3	(16)	12.3	(7)	3.0	(2)	5.3	31.9	−13.625
	(12)	12.3	(13)	8.7	(6)	8.0	(3)	9.3	38.3	− 7.225
	(11)	30.3	(14)	22.3	(5)	11.0	(4)	12.7	76.3	30.775
Column totals		70.9		50.3		32.3		28.6	182.1	0
$Y_{3\cdot i\cdot}-3\bar{y}_{3\cdots}$		25.375		4.775		−13.225		−16.925	0	
Square IV										
	(16)	5.0	(12)	10.3	(8)	5.7	(4)	12.7	33.7	− 9.750
	(11)	2.7	(15)	6.7	(3)	10.3	(7)	5.7	25.4	−18.050
	(1)	1.0	(5)	10.3	(9)	11.3	(13)	11.7	34.3	− 9.150
	(6)	11.0	(2)	19.0	(14)	20.7	(10)	29.7	80.4	36.950
Column totals		19.7		46.3		48.0		59.8	173.8	0
$Y_{4\cdot i\cdot}-4\bar{y}_{4\cdots}$		−23.750		2.850		4.550		16.350	0	
Square V										
	(3)	2.0	(16)	5.0	(5)	4.0	(10)	13.7	24.7	−22.575
	(6)	9.3	(9)	1.7	(4)	6.3	(15)	12.3	29.6	−17.675
	(12)	16.7	(7)	4.3	(14)	18.7	(1)	8.7	48.4	1.125
	(13)	16.7	(2)	30.0	(11)	25.7	(8)	14.0	86.4	39.125
Column totals		44.7		41.0		54.7		48.7	189.1	0
$Y_{5\cdot i\cdot}-4\bar{y}_{5\cdots}$		−2.575		−6.275		7.425		1.425	0	872.4

Table 3. ANOVA for data in Table 2.

Source of variation	d.f.	Sum of squares	Mean square
Total	80	13,122.06	—
Correction for mean	1	$872.4^2/80 =$ 9,513.522	
Replicate	4	31.56	—
Treatment ignoring row and column, TssiRC	15	1,244.202	—
Row eliminating treatment ignoring column, RsseTiC	15	1,093.016	—
Column eliminating treatment and row, CsseTR	15	559.590	37.306
Intrarow-column error	30	680.170	22.672
Row eliminating treatment and column, RsseTC	15	1,026.756	68.450
Column eliminating treatment ignoring row, CsseTiR	15	625.849	—
Treatment eliminating row and column, TsseRC	15	319.452	21.30

$\hat{\sigma}_\epsilon^2 = \mathrm{E} = 22.672$, $\hat{\sigma}_\beta^2 = \mathrm{R} = 15.259$, $\hat{\sigma}_\gamma^2 = \mathrm{C} = 4.878$.

$\mathrm{c} + \mathrm{E/R} = 5.486$, $\mathrm{k} + \mathrm{E/C} = 8.648$.

Replacing c and k with c + E/R and k + E/C in (3.2), respectively, results in treatment effects recovering row and column information TeRCr. These values plus the general mean, 872.4/80 = 10.905 result in the adjusted means given in Table 12.6 of Cochran and Cox (1957).

Since this is a balanced lattice square, all variances of differences between two treatment effects recovering row and column information are the same, i.e., 11.907. The variance-covariance matrix is

$$\text{varcov} = \left[\mathrm{r}\mathbf{I}_v - \mathbf{RT}'\left(\mathbf{RT} - \mathbf{J}_{rk\times v}\right)\Big/(\mathrm{c} + \mathrm{E/R}) - \mathbf{CT}'\left(\mathbf{CT} - \mathbf{J}_{rc\times v}\right)\Big/(\mathrm{k} + \mathrm{E/C})\right]^{-1} . \tag{3.5}$$

Then, two times the 1/vth root of the determinant of varcov results in the correct average variance of a difference, 11.907, as the conditions for equality are satisfied.

4. GAUSS Program For an IBED With GRADIENTS Within Blocks

Federer (1994) has presented a method for analyzing incomplete block or lattice rectangle experiment designs when there are differential gradients or trends within an incomplete block or within a row of the experiment design. A GAUSS program for this analysis is given in Table A3. Here the linear model used is

$$\mathrm{Y}_{ghi} = \mu + \rho_g + \beta_{gh} + \gamma_{gh}\,\mathrm{a}_{ghi} + \tau_i + \epsilon_{ghi} , \tag{4.1}$$

where γ_{gh} is a linear regression coefficient on order, a_{ghi}, within an incomplete block or row and the remaining effects are as for (2.1). The data set used to illustrate the analysis using GAUSS is that of Table 2. The treatment-block(row)-gradient incidence matrix is

$$\begin{bmatrix} \mathrm{k}\mathbf{I}_{rb} & \mathbf{0}_{rb\times rb} & \mathbf{RT}_{rb\times v} \\ \mathbf{0}_{rb\times rb} & \sum_h \mathrm{a}^2_{ghi}\mathbf{I}_{rb} & \mathbf{TG}_{rb\times v} \\ \mathbf{RT}'_{v\times rb} & \mathbf{TG}'_{v\times rb} & \mathrm{r}\mathbf{I}_v \end{bmatrix} \begin{bmatrix} \mathbf{BeTG} \\ \mathbf{GeTB} \\ \mathbf{TeBG} \end{bmatrix} = \begin{bmatrix} \mathbf{YB} \\ \mathbf{YG} \\ \mathbf{YT} \end{bmatrix}, \tag{4.2}$$

where $\mathbf{0}_{rb\times rb}$ is a matrix of zeros since $\sum_h \mathrm{a}_{ghi} = 0$ for every block, **RT** is the same as **NB** in Section 2, and **YG** is a rb × 1 vector of sums of products of responses Y_{ghi} and order. Here there are four experimental units in each row and we assign −3, −1, 1, and 3 to the four positions in order. The first entry in **YG** is computed as

$$-3(9.0) - 1(20.3) + 1(17.7) + 3(26.3) = 49.3 .$$

The sum of squares of coefficients is $(-3)^2 + (-1)^2 + 1^2 + 3^2 = 20 = \text{cx}$. The remaining 19 sums of products in **YG** are computed similarly.

The various sums of squares are obtained from Table 3 and Table A3 and are summarized in Table 4. The variance-covariance matrix is

$$\text{varcov} = \text{E}\left[\text{r}\mathbf{I}_v - \mathbf{RT}'\left(\mathbf{RT} - \mathbf{J}_{rc\times v}\right)\Big/ 5.3360 - \mathbf{GT}'\left(\mathbf{GT} - \mathbf{J}_{rc\times v}\right)\Big/ 34.7413\right]^{-1} . \tag{4.3}$$

An approximate average variance of a difference between two treatment effects with recovery of interrow and intergradient information is two times the 1/vth root of the determinant of varcov.

The expected value of the gradient eliminating treatment and row mean square is

$$\sigma_\epsilon^2 + \sum_h \text{a}_{ghi}^2(\text{rk}-\text{k}-1)\sigma_\gamma^2 \Big/ \text{rc} . \tag{4.4}$$

The expected value of the row eliminating treatment and gradient mean square is

$$\sigma_\epsilon^2 + 2.8149\sigma_\beta^2 , \tag{4.5}$$

where the coefficient 2.8149 was obtained using the MATHEMATICA program in Table A4 and as described by Federer and Hsu-Schmidt (1995). It was not possible to obtain the expected value of RsseTG using SAS-GLM. To date the SAS office has not responded to a query concerning this. The algebraic formulation appears complex.

Note: Using ad = d^(1/v) did not work on my computer. I used ad = d^(1/16) and it worked to obtain an approximate average variance of a difference of 10.890, which is less than the one obtained for the lattice square analysis.

Table 4. ANOVA for trends within rows of the design in Table 2.

Source of variation	d.f.	Sum of squares	Mean square
Total	80	13,122.06	—
Correction for mean	1	9,513.522	—
Replicate	4	31.56	—
Treatment ignoring row and column, TssiRG	15	1,244.202	—
Row eliminating treatment ignoring gradient, RsseTiG	15	1,093.016	—
Gradient eliminating row and treatment, GsseTR	20	765.496	38.2748
Intrarow-gradient error	25	474.264	18.9706
Row eliminating treatment and gradient, RsseTG	15	884.113	58.9409
Gradient ignoring row eliminating treatment, GsseTiR	20	974.399	—
Treatment eliminating row and gradient, TsseRG	15	347.188	23.1459

$\hat{\sigma}_\epsilon^2 = \mathrm{E} = 18.9706$, $\hat{\sigma}_\beta^2 = \mathrm{B} = 14.1995$, $\hat{\sigma}_\gamma^2 = \mathrm{G} = 1.2869$.

$\mathrm{c} + \mathrm{E/B} = 5.3360$, $\sum_h \mathrm{a}_{ghi}^2 + \mathrm{E/G} = 34.7413$.

5. Literature Cited

Cochran, W.G. and G.M. Cox (1957). *Experimental Designs*, 2nd edition. John Wiley and Sons, Inc., New York.

Federer, W.T. (1955). *Experimental Design — Theory and Application.* Macmillan, New York. (Republished by Oxford and IBH Publishing Company, New Delhi, 1967 and 1974.)

Federer, W.T. (1994). Recovery of intergradient and interblock information in incomplete block and lattice rectangle designed experiments. BU-1265-M in the Technical Report Series of the Biometrics Unit, Cornell University, Ithaca, NY.

Federer, W.T. and S.F. Hsu-Schmitz (1995). MATHEMATICA for expected values of sums of squares. BU- -M in the Technical Report Series of the Biometrics Unit, Cornell University, Ithaca, NY.

WTF:np

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@Table A1.  This is a GAUSS program for an incomplete block design with v
treatments in b incomplete blocks of size k in r complete blocks.  E is the
intrablock error component of variance and B is the interblock component of
variance.  The design under consideration to illustrate the procedure is
    1   2   3            1   4   7            1   5   9
    4   5   6            2   5   8            2   6   7
    7   8   9            3   6   9            3   4   8
The rv by 1 vector of responses is denoted as Y[rv,1].@
r = 3; v = 9; b = 3; k = 3;"Y[27,1] =";
let Y[27,1] = 6 8 7  12 10 11  12 12 12  9 9 9  7 11 9  11 13 12
9 11 10  10 12 11  8 10 9;format 2,0; Y';
"The design matrix is X[rv,(1+r+rb+v)] ="; let X[27,22] =
1 1 0 0  1 0 0 0 0 0 0 0 0  1 0 0 0 0 0 0 0 0
1 1 0 0  1 0 0 0 0 0 0 0 0  0 1 0 0 0 0 0 0 0
1 1 0 0  1 0 0 0 0 0 0 0 0  0 0 1 0 0 0 0 0 0
1 1 0 0  0 1 0 0 0 0 0 0 0  0 0 0 1 0 0 0 0 0
1 1 0 0  0 1 0 0 0 0 0 0 0  0 0 0 0 1 0 0 0 0
1 1 0 0  0 1 0 0 0 0 0 0 0  0 0 0 0 0 1 0 0 0
1 1 0 0  0 0 1 0 0 0 0 0 0  0 0 0 0 0 0 1 0 0
1 1 0 0  0 0 1 0 0 0 0 0 0  0 0 0 0 0 0 0 1 0
1 1 0 0  0 0 1 0 0 0 0 0 0  0 0 0 0 0 0 0 0 1

1 0 1 0  0 0 0 1 0 0 0 0 0  1 0 0 0 0 0 0 0 0
1 0 1 0  0 0 0 1 0 0 0 0 0  0 0 0 1 0 0 0 0 0
1 0 1 0  0 0 0 1 0 0 0 0 0  0 0 0 0 0 0 1 0 0
1 0 1 0  0 0 0 0 1 0 0 0 0  0 1 0 0 0 0 0 0 0
1 0 1 0  0 0 0 0 1 0 0 0 0  0 0 0 0 1 0 0 0 0
1 0 1 0  0 0 0 0 1 0 0 0 0  0 0 0 0 0 0 0 1 0
1 0 1 0  0 0 0 0 0 1 0 0 0  0 0 1 0 0 0 0 0 0
1 0 1 0  0 0 0 0 0 1 0 0 0  0 0 0 0 0 1 0 0 0
1 0 1 0  0 0 0 0 0 1 0 0 0  0 0 0 0 0 0 0 0 1

1 0 0 1  0 0 0 0 0 0 1 0 0  1 0 0 0 0 0 0 0 0
1 0 0 1  0 0 0 0 0 0 1 0 0  0 0 0 0 1 0 0 0 0
1 0 0 1  0 0 0 0 0 0 1 0 0  0 0 0 0 0 0 0 0 1
1 0 0 1  0 0 0 0 0 0 0 1 0  0 1 0 0 0 0 0 0 0
1 0 0 1  0 0 0 0 0 0 0 1 0  0 0 0 0 0 1 0 0 0
1 0 0 1  0 0 0 0 0 0 0 1 0  0 0 0 0 0 0 1 0 0
1 0 0 1  0 0 0 0 0 0 0 0 1  0 0 1 0 0 0 0 0 0
1 0 0 1  0 0 0 0 0 0 0 0 1  0 0 0 1 0 0 0 0 0
1 0 0 1  0 0 0 0 0 0 0 0 1  0 0 0 0 0 0 0 1 0;
"The various totals are"; tot = X'*Y; tot';
"The total sum of squares is"; tss = Y'*Y; tss;
"The rb by 1 vector of incomplete block totals minus k times the replicate
mean is YB[rb,1]"; let YB[9,1] = -9 3 6  -3 -3 6  0 3 -3;
"The sum of squares for incomplete blocks ignoring treatments = BssiT.";
BssiT = YB'*YB/k; BssiT;
"The v by 1 vector of treatment totals minus r times grand mean = YT[v,1].";
let YT[9,1] = -6 -5 -4  1 2 6  2 0 4;
"The sum of squares for treatments ignoring blocks = TssiB";
TssiB = YT'*YT/r;  TssiB;
"The incomplete block by treatment incidence matrix is NB[v,rb].";let NB[9,9] =
1 0 0   1 0 0   1 0 0
1 0 0   0 1 0   0 1 0
1 0 0   0 0 1   0 0 1
0 1 0   1 0 0   0 0 1
0 1 0   0 1 0   1 0 0
0 1 0   0 0 1   0 1 0
0 0 1   1 0 0   0 1 0
0 0 1   0 1 0   0 0 1
0 0 1   0 0 1   1 0 0;
```

```
Irb = eye(9); Iv = eye(9); Jv = ones(9,9); Jk = ones(3,3);
Ir = eye(3); Jb = Ir.*.Jk;
"The incomplete block effects eliminating treatments is BeT.";
BeT = inv(k*Irb - NB'*NB/r + Jb/r)*(YB - NB'*YT/r); BeT';
"Incomplete block eliminating treatment sum of squares = BsseT.";
BsseT = BeT'*(YB - NB'*YT/r); BsseT;
"Treatment effects eliminating incomplete blocks is TeB.";
TeB = inv(r*Iv - NB*NB'/k + Jv/k)*(YT - NB*YB/k); TeB';
"Treatment eliminating incomplete block sum of squares = TsseB.";
TsseB = TeB'*(YT - NB*YB/k); TsseB;
E = 0.8; B = 2.667;  format 2,2;
"Treatment effects with recovery of interblock information = TeBr.";
TeBr = inv(r*Iv - NB*NB'/(k + E/B) + Jv/(k + E/B))*(YT - NB*YB/(k + E/B));
TeBr';
varcov = E*inv(r*Iv - NB*NB'/(k + E/B) + Jv/(k + E/B)); varcov;
d = det(varcov); ad = d^(1/v);
"Approximate average variance of a difference = vardif.";
vardif = 2*ad; vardif;
```

```
Executing: D:\GAUSS\INCBLOCK.933
Y[27,1] =
 6  8  7 12 10 11 12 12 12  9  9  9  7 11  9 11 13 12  9 11 10 10 12 11  8 10
 9
The design matrix is X[rv,(1+r+rb+v)] =
The various totals are
270 90 90 90 21 33 36 27 27 36 30 33 27 24 25 26 31 32 36 32 30 34
The total sum of squares is
2786
The rb by 1 vector of incomplete block totals minus k times the replicate
mean is YB[rb,1]
The sum of squares for incomplete blocks ignoring treatments = BssiT.
66
The v by 1 vector of treatment totals minus r times grand mean = YT[v,1].
The sum of squares for treatments ignoring blocks = TssiB
46
The incomplete block by treatment incidence matrix is NB[v,rb].
The incomplete block effects eliminating treatments is BeT.
-2 -5E-017  2 -1 -1  2 -6E-017  1 -1
Incomplete block eliminating treatment sum of squares = BsseT.
32
Treatment effects eliminating incomplete blocks is TeB.
-1 -1 -1  1  1  1 1E-017 1E-017 1E-017
Treatment eliminating incomplete block sum of squares = TsseB.
12
Treatment effects with recovery of interblock information = TeBr.
-1.13 -1.09 -1.04 0.91 0.96 1.13 0.09 -1.64E-017 0.17

0.34 0.00 0.00 0.00 0.00 -0.04 0.00 -0.04 0.00
0.00 0.34 0.00 -0.04 0.00 0.00 0.00 0.00 -0.04
0.00 0.00 0.34 0.00 -0.04 0.00 -0.04 0.00 0.00
0.00 -0.04 0.00 0.34 0.00 0.00 0.00 0.00 -0.04
0.00 0.00 -0.04 0.00 0.34 0.00 -0.04 0.00 0.00
-0.04 0.00 0.00 0.00 0.00 0.34 0.00 -0.04 0.00
0.00 0.00 -0.04 0.00 -0.04 0.00 0.34 0.00 0.00
-0.04 0.00 0.00 0.00 0.00 -0.04 0.00 0.34 0.00
0.00 -0.04 0.00 -0.04 0.00 0.00 0.00 0.00 0.34
Approximate average variance of a difference = vardif.
0.68
```

```
@Table A2.  GAUSS program for analyzing a lattice square or lattice rectangle
design for v = kc treatments in k rows and c columns in r complete blocks.
The data used to illustrate the program are given in Table 12.5 of Cochran
and Cox, 1957.@
r = 5; v = 16; k = 4; c = 4;format 2,3;
" The rv by 1 vector of observations, Y[rv,1], is";
"The design matrix X[rv,(1+r+rk+rc+v) is";
"The various totals are obtained as tot = X'*Y";
"The rk row totals minus c times the complete block mean are ";
let YR[20,1] = 32.475 -11.525 -9.125 -11.825 15.375 4.275 -7.325 -12.325
-9.925 -13.625 -7.225 30.775 -9.750 -18.050 -9.150 36.950
-22.575 -17.675 1.125 39.125;
"The row ignoring treatment and column sum of squares RssiTC is";
RssiTC = YR'*YR/c; RssiTC;
"The rc by 1 column totals minus k times the complete block mean are";
let YC[20,1] = -14.125 0.175 1.575 12.375 6.375 -16.325 -3.025 12.975 25.375
4.775 -13.225 -16.925 -23.750 2.850 4.550 16.350 -2.575 -6.275 7.425 1.425;
"The column ignoring row and treatment sum of squares CssiTR is";
CssiTR = YC'*YC/k; CssiTR;
"The v by 1 treatment totals minus r times the grand mean are";
let YT[16,1] = -29.925 20.175 -20.925 1.475 -4.525 -20.525 -21.225 -6.225
-4.525 26.875 43.475 13.775 6.275 19.875 -11.225 -12.825;
"The treatment ignoring row and column sum of squares TssiRC is";
TssiRC = YT'*YT/r; TssiRC;
Irk = eye(20); Irc = eye(20); Iv = eye(16); Ir = eye(5);
Jv = ones(16,16); Jrv = ones(20,16); Jcv = ones(20,16); let RT[20,16] =
0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0

0 1 0 0 1 0 0 0 0 0 0 1 0 0 1 0
0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0
1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1
0 0 1 0 0 0 0 1 1 0 0 0 0 1 0 0

1 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0
0 1 0 0 0 0 1 0 1 0 0 0 0 0 0 1
0 0 1 0 0 1 0 0 0 0 0 1 1 0 0 0
0 0 0 1 1 0 0 0 0 0 1 0 0 1 0 0

0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0

0 0 1 0 1 0 0 0 0 1 0 0 0 0 0 1
0 0 0 1 0 1 0 0 1 0 0 0 0 0 1 0
1 0 0 0 0 0 1 0 0 0 0 1 0 1 0 0
0 1 0 0 0 0 0 1 0 0 1 0 1 0 0 0;
let CT[20,16] =
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
```

```
0 0 1 0 1 0 0 0 0 1 0 0 0 0 0 1
1 0 0 0 0 0 1 0 0 0 0 1 0 1 0 0
0 0 0 1 0 1 0 0 1 0 0 0 0 0 1 0
0 1 0 0 0 0 0 1 0 0 1 0 1 0 0 0

0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0

1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1
0 1 0 0 1 0 0 0 0 0 0 1 0 0 1 0
0 0 1 0 0 0 0 1 1 0 0 0 0 1 0 0
0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0

0 0 1 0 0 1 0 0 0 0 0 1 1 0 0 0
0 1 0 0 0 0 1 0 1 0 0 0 0 0 0 1
0 0 0 1 1 0 0 0 0 0 1 0 0 1 0 0
1 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0; format 3,3; Z = zeros(20,20);
"Row eliminating treatment ignoring column effects ReTiC are";
ReTiC = inv(c*Irk - (RT - Jrv)*RT'/r)*(YR - RT*YT/r);
"Row eliminating treatment ignoring column sum of squares RsseTiC is";
RsseTiC = ReTiC'*(YR - RT*YT/r); RsseTiC;
"Column eliminating treatment ignoring row effects CeTiR are";
CeTiR = inv(k*Irc - (CT - Jcv)*CT'/r)*(YC - CT*YT/r);
"Column eliminating treatment ignoring row sum of squares CsseTiR is";
CsseTiR = CeTiR'*(YC - CT*YT/r); CsseTiR;
"Row eliminating both treatment and column effects ReTC are";
ReTC = inv(c*Irk - (Z~RT)*inv(k*Irc~(CT - Jcv)¦(CT'~(r*Iv)))*(Z'¦RT'))*
(YR - (Z~RT)*inv(k*Irc~(CT - Jcv)¦(CT'~r*Iv))*(YC¦YT));
"Row eliminating both treatment and column sum of squares RsseTC is";
RsseTC = ReTC'*(YR - (Z~RT)*inv(k*Irc~(CT - Jcv)¦(CT'~r*Iv))*(YC¦YT));RsseTC;
"Column eliminating both treatment and row effects CeTR are";
CeTR = inv(k*Irc - (Z~CT)*inv(c*Irk~(RT - Jrv)¦(RT'~r*Iv))*(Z'¦CT'))*
(YC - (Z~CT)*inv(k*Irk~(RT - Jrv)¦RT'~r*Iv)*(YR¦YT));
"Column eliminating both treatment and row sum of squares CsseTR is";
CsseTR = CeTR'*(YC - (Z~CT)*inv(c*Irk~(RT - Jrv)¦(RT'~r*Iv))*(YR¦YT));CsseTR;
"Intra row and column treatment effects TeRC are";
TeRC = inv(r*Iv - RT'*RT/c - CT'*CT/k + Jv/k + Jv/c)*(YT - RT'*YR/c -CT'*YC/k);
"Treatment eliminating both row and column effects TeRCr and recovering
row and column information are";
TeRCr = inv(r*Iv - (RT'*RT - Jv)/5.486 - (CT'*CT - Jv)/8.648 )*
(YT - RT'*YR/5.486 - CT'*YC/8.648); TeRCr';
"Treatment eliminating both row and column sum of squares TsseRC is";
TsseRC = TeRC'*(YT - RT'*YR/c - CT'*YC/k); TsseRC;
"Variance-covariance matrix for TeRCr is varcov";
varcov = 22.672*inv(r*Iv - (RT'*RT - Jv)/5.486 - (CT'*CT - Jv)/8.648);
"Approximate average variance of a difference is vardif";
vardif = 2*det(varcov)^(1/v);vardif;


D:\GAUSS>type incblock.933
```

```
Executing: D:\GAUSS\LATTSQ16.45
 The rv by 1 vector of observations, Y[rv,1], is
The design matrix X[rv,(1+r+rk+rc+v) is
The various totals are obtained as tot = X'*Y
The rk row totals minus c times the complete block mean are
The row ignoring treatment and column sum of squares RssiTC is
1844.545
The rc by 1 column totals minus k times the complete block mean are
The column ignoring row and treatment sum of squares CssiTR is
732.810
The v by 1 treatment totals minus r times the grand mean are
The treatment ignoring row and column sum of squares TssiRC is
1244.202
Row eliminating treatment ignoring column effects ReTiC are
Row eliminating treatment ignoring column sum of squares RsseTiC is
1093.016
Column eliminating treatment ignoring row effects CeTiR are
Column eliminating treatment ignoring row sum of squares CsseTiR is
625.849
Row eliminating both treatment and column effects ReTC are
Row eliminating both treatment and column sum of squares RsseTC is
1026.756
Column eliminating both treatment and row effects CeTR are
Column eliminating both treatment and row sum of squares CsseTR is
559.590
Intra row and column treatment effects TeRC are
Treatment eliminating both row and column effects TeRCr and recovering
row and column information are
-4.457 2.777 -2.178 0.453 -1.464 -3.328 -3.534 -1.589 -0.891 4.010 6.686 1.797
-0.214 3.366 -1.620 0.187
Treatment eliminating both row and column sum of squares TsseRC is
319.452
Variance-covariance matrix for TeRCr is varcov
Approximate average variance of a difference is vardif
11.907
```

```
@Table A3.  GAUSS program for analyzing an incomplete block or lattice
rectangle design with a linear trend within the block or row for v = kc
treatments in c blocks of size k in r complete blocks.  The data used to
illustrate the analysis is the lattice square design in Table 12.5 of
Cochran and Cox (1957).@
r = 5; v = 16; k = 4; c = 4; format 2,3;
"The rc row totals minus k times the complete block mean are";
let YR[20,1] = 32.475 -11.525 -9.125 -11.825 15.375 4.275 -7.325 -12.325
-9.925 -13.625 -7.225 30.775 -9.750 -18.050 -9.150 36.950
-22.575 -17.675 1.125 39.125;
"Row ignoring treatment and gradient sum of squares RssiTG is";
RssiTG = YR'*YR/c; RssiTG;
"Sum of products for gradients in each block ignoring row and treatment is";
let YG[20,1] = 49.3 9.1 -9.1 31.6 -5.6 9.3 -3.7 33.1 -43.8 -27.3 -9.7 -64.1
18.5 12.6 33.1 57.8 34.1 13.6 -9.6 -12.4; cx = 20;
"Gradient ignoring treatment and row sum of squares GssiTR is";
GssiTR = YG'*YG/cx; GssiTR;
"The v treatment totals minus r times the grand mean is";
let YT[16,1] = -29.925 20.175 -20.925 1.475 -4.525 -20.525 -21.225 -6.225
-4.525 26.875 43.475 13.775 6.275 19.875 -11.225 -12.825;
"Treatment ignoring row and gradient sum of squares TssiRG is";
TssiRG = YT'*YT/r; TssiRG;
Irk = eye(20); Irg = eye(20); Iv = eye(16); Z = zeros(20,20);
Jrv = ones(20,16); Jv = ones(16,16); let RT[20,16] =
0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0

0 1 0 0 1 0 0 0 0 0 0 1 0 0 1 0
0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0
1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1
0 0 1 0 0 0 0 1 1 0 0 0 0 1 0 0

1 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0
0 1 0 0 0 0 1 0 1 0 0 0 0 0 0 1
0 0 1 0 0 1 0 0 0 0 0 1 1 0 0 0
0 0 0 1 1 0 0 0 0 0 1 0 0 1 0 0

0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0

0 0 1 0 1 0 0 0 0 1 0 0 0 0 0 1
0 0 0 1 0 1 0 0 1 0 0 0 0 0 1 0
1 0 0 0 0 0 1 0 0 0 0 1 0 1 0 0
0 1 0 0 0 0 0 1 0 0 1 0 1 0 0 0;
 let TG[20,16] =
 0 0 0 0 0 0 0 0 1 -3 3 -1 0 0 0 0
 1 -3 3 -1 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 0 0 0 1 -3 3 -1
 0 0 0 0 1 -3 3 -1 0 0 0 0 0 0 0 0

 0 3 0 0 -3 0 0 0 0 0 0 -1 0 0 1 0
 0 0 0 1 0 0 -1 0 0 -3 0 0 3 0 0 0
 -1 0 0 0 0 1 0 0 0 0 3 0 0 0 0 -3
 0 0 -3 0 0 0 0 3 1 0 0 0 0 -1 0 0
```

```
3 0 0 0 0 0 0 1 0 -3 0 0 0 0 -1 0
0 3 0 0 0 0 1 0 -3 0 0 0 0 0 0 -1
0 0 3 0 0 1 0 0 0 0 0 -3 -1 0 0 0
0 0 0 3 1 0 0 0 0 0 -3 0 0 -1 0 0

0 0 0 3 0 0 0 1 0 0 0 -1 0 0 0 -3
0 0 1 0 0 0 3 0 0 0 -3 0 0 0 -1 0
-3 0 0 0 -1 0 0 0 1 0 0 0 3 0 0 0
0 -1 0 0 0 -3 0 0 0 3 0 0 0 1 0 0

0 0 -3 0 1 0 0 0 0 3 0 0 0 0 0 -1
0 0 0 1 0 -3 0 0 -1 0 0 0 0 0 3 0
3 0 0 0 0 0 -1 0 0 0 0 -3 0 1 0 0
0 -1 0 0 0 0 0 3 0 0 1 0 -3 0 0 0;format 3,3;
ReTiG = inv(c*Irk - (RT - Jrv)*RT'/r)*(YR - RT*YT/r);
"Row eliminating treatment ignoring gradient sum of squares RsseTiG is";
RsseTiG = ReTiG'*(YR - RT*YT/r); RsseTiG;
GeTiR = inv(cx*Irg - TG*TG'/r)*(YG - TG*YT/r);
"Gradient eliminating treatment ignoring row sum of squares GsseTiR is";
GsseTiR = GeTiR'*(YG - TG*YT/r); GsseTiR;
V = c*Irk~(RT - Jrv)¦RT'~r*Iv;
U = cx*Irg~TG¦TG'~r*Iv;
QT = YT - RT'*YR/c - TG'*YG/cx;
rx = YR - (Z~RT)*inv(U)*(YG¦YT);
gx = YG - (Z~TG)*inv(V)*(YR¦YT);
ReTG = inv(c*Irk - (Z~(RT - Jrv))*inv(U)*(Z¦RT'))*rx;
"Row eliminating both treatment and gradient sum of squares RsseTG is";
RsseTG = ReTG'*rx; RsseTG;
GeTR = inv(cx*Irg - (Z~TG)*inv(V)*(Z¦TG'))*gx;
"Gradient eliminating both treatment and row sum of squares GsseTR is";
GsseTR = GeTR'*gx; GsseTR;
TeRG = inv(r*Iv - RT'*RT/k - TG'*TG/cx + Jv/k)*QT;
"Treatment eliminating both row and gradient sum of squares TsseRG is";
TsseRG = TeRG'*QT; TsseRG;
"Treatment effects with recovery of interrow and intergradient information
TeRGr are";
TeRGr = inv(r*Iv - (RT'*RT - Jv)/5.3360 - TG'*TG/34.7413)*(YT -
RT'*YR/5.3360 - TG'*YG/34.7413);TeRGr';
"Variance-covariance matrix varcov is"; E = 18.9706;
varcov = E*inv(r*Iv - (RT'*RT - Jv)/5.3360 - TG'*TG/34.7413); varcov;
"Approximate average variance of a difference vardif is";
d = det(varcov); d;
ad = d^(1/16);ad;
vardif = 2*ad; vardif;
```

```
Executing: D:\GAUSS\LSGRAD16.45
The rc row totals minus k times the complete block mean are
Row ignoring treatment and gradient sum of squares RssiTG is
1844.545
Sum of products for gradients in each block ignoring row and treatment is
Gradient ignoring treatment and row sum of squares GssiTR is
910.980
The v treatment totals minus r times the grand mean is
Treatment ignoring row and gradient sum of squares TssiRG is
1244.202
Row eliminating treatment ignoring gradient sum of squares RsseTiG is
1093.016
Gradient eliminating treatment ignoring row sum of squares GsseTiR is
974.399
Row eliminating both treatment and gradient sum of squares RsseTG is
884.113
Gradient eliminating both treatment and row sum of squares GsseTR is
765.496
Treatment eliminating both row and gradient sum of squares TsseRG is
347.188
Treatment effects with recovery of interrow and intergradient information
TeRGr are
-5.324 3.852 -1.784 1.138 -1.500 -2.749 -4.596 -1.538 -1.632 4.438 6.594 2.381
-0.217 2.210 -0.618 -0.656
Variance-covariance matrix varcov is

5.697 -0.169 0.253 -0.048 0.121 0.019 -0.135 0.158 -0.106 -0.490 -0.096 -0.437
-0.440 0.154 -0.177 0.161
-0.169 5.680 -0.428 0.164 -0.378 0.123 0.111 -0.116 -0.367 -0.132 -0.032 -0.103
0.192 -0.075 0.164 -0.171
0.253 -0.428 6.148 -0.166 -0.120 0.177 0.116 -0.409 -0.105 -0.538 -0.126 -0.452
-0.139 0.165 -0.107 0.195
-0.048 0.164 -0.166 5.291 0.113 -0.133 -0.010 0.121 -0.059 -0.100 -0.394 -0.104
0.159 -0.154 0.152 -0.367
0.121 -0.378 -0.120 0.113 4.952 -0.166 0.142 -0.021 -0.030 0.202 -0.176 0.138
-0.149 -0.025 -0.118 -0.021
0.019 0.123 0.177 -0.133 -0.166 5.701 -0.434 0.163 0.156 -0.580 0.257 -0.183
-0.021 -0.100 -0.395 -0.121
-0.135 0.111 0.116 -0.010 0.142 -0.434 5.303 -0.164 -0.157 0.259 -0.481 0.157
-0.121 -0.026 -0.096 -0.001
0.158 -0.116 -0.409 0.121 -0.021 0.163 -0.164 5.295 0.136 -0.142 0.194 -0.051
-0.389 -0.118 -0.048 -0.147
-0.106 -0.367 -0.105 -0.059 -0.030 0.156 -0.157 0.136 4.950 -0.171 0.167 -0.024
0.123 -0.034 -0.129 0.113
-0.490 -0.132 -0.538 -0.100 0.202 -0.580 0.259 -0.142 -0.171 6.695 -0.536 0.260
-0.441 0.121 0.173 -0.117
-0.096 -0.032 -0.126 -0.394 -0.176 0.257 -0.481 0.194 0.167 -0.536 6.145 -0.173
-0.112 0.112 0.118 -0.405
-0.437 -0.103 -0.452 -0.104 0.138 -0.183 0.157 -0.051 -0.024 0.260 -0.173 5.303
0.157 -0.130 -0.014 0.119
-0.440 0.192 -0.139 0.159 -0.149 -0.021 -0.121 -0.389 0.123 -0.441 -0.112 0.157
5.688 -0.162 0.161 -0.045
0.154 -0.075 0.165 -0.154 -0.025 -0.100 -0.026 -0.118 -0.034 0.121 0.112 -0.130
-0.162 4.949 -0.353 0.138
-0.177 0.164 -0.107 0.152 -0.118 -0.395 -0.096 -0.048 -0.129 0.173 0.118 -0.014
0.161 -0.353 5.292 -0.160
0.161 -0.171 0.195 -0.367 -0.021 -0.121 -0.001 -0.147 0.113 -0.117 -0.405 0.119
-0.045 0.138 -0.160 5.293
Approximate average variance of a difference vardif is
5.973E+011
5.445
10.890
```

Table A4. MATHEMATICA program for expected values of sums of squares.

```
S1 = Sum[b[1,h],{h,1,4}];S2 = Sum[b[2,h],{h,1,4}];
S3 = Sum[b[3,h],{h,1,4}];S4 = Sum[b[4,h],{h,1,4}];
S5 = Sum[b[5,h],{h,1,4}];
S = Sum[b[g,h],{g,1,5},{h,1,4}]/4;r = 5;k = 4; v = 16;
b11 = 4 b[1,1] +  t[9] + t[10] + t[11] + t[12] - S1;
b12 = 4 b[1,2] + t[1] + t[2] + t[3] + t[4] - S1;
b13 = 4 b[1,3] + t[13] + t[14] + t[15] + t[16] - S1;
b14 = 4 b[1,4] + t[5] + t[6] + t[7] + t[8] - S1;
b21 = 4 b[2,1] + t[2] + t[5] + t[12] + t[15] - S2;
b22 = 4 b[2,2] + t[4] + t[7] + t[10] + t[13] - S2;
b23 = 4 b[2,3] + t[1] + t[6] + t[11] + t[16] - S2;
b24 = 4 b[2,4] + t[3] + t[8] + t[9] + t[14] - S2;
b31 = 4 b[3,1] + t[1] + t[8] + t[10] + t[15] - S3;
b32 = 4 b[3,2] + t[2] + t[7] + t[9] + t[16] - S3;
b33 = 4 b[3,3] + t[3] + t[6] + t[12] + t[13] - S3;
b34 = 4 b[3,4] + t[4] + t[5] + t[11] + t[14] - S3;
b41 = 4 b[4,1] + t[4] + t[8] + t[12] + t[16] - S4;
b42 = 4 b[4,2] + t[3] + t[7] + t[11] + t[15] - S4;
b43 = 4 b[4,3] + t[1] + t[5] + t[9] + t[13] - S4;
b44 = 4 b[4,4] + t[2] + t[6] + t[10] + t[14] - S4;
b51 = 4 b[5,1] + t[3] + t[5] + t[10] + t[16] - S5;
b52 = 4 b[5,2] + t[4] + t[6] + t[9] + t[15] - S5;
b53 = 4 b[5,3] + t[1] + t[7] + t[12] + t[14] - S5;
b54 = 4 b[5,4] + t[2] + t[8] + t[11] + t[13] - S5;NB =
{{0,0,0,0,  0,0,0,0,  1,1,1,1,  0,0,0,0},
 {1,1,1,1,  0,0,0,0,  0,0,0,0,  0,0,0,0},
 {0,0,0,0,  0,0,0,0,  0,0,0,0,  1,1,1,1},
 {0,0,0,0,  1,1,1,1,  0,0,0,0,  0,0,0,0},
 {0,1,0,0,  1,0,0,0,  0,0,0,1,  0,0,1,0},
 {0,0,0,1,  0,0,1,0,  0,1,0,0,  1,0,0,0},
 {1,0,0,0,  0,1,0,0,  0,0,1,0,  0,0,0,1},
 {0,0,1,0,  0,0,0,1,  1,0,0,0,  0,1,0,0},
 {1,0,0,0,  0,0,0,1,  0,1,0,0,  0,0,1,0},
 {0,1,0,0,  0,0,1,0,  1,0,0,0,  0,0,0,1},
 {0,0,1,0,  0,1,0,0,  0,0,0,1,  1,0,0,0},
 {0,0,0,1,  1,0,0,0,  0,0,1,0,  0,1,0,0},
 {0,0,0,1,  0,0,0,1,  0,0,0,1,  0,0,0,1},
 {0,0,1,0,  0,0,1,0,  0,0,1,0,  0,0,1,0},
 {1,0,0,0,  1,0,0,0,  1,0,0,0,  1,0,0,0},
 {0,1,0,0,  0,1,0,0,  0,1,0,0,  0,1,0,0},
 {0,0,1,0,  1,0,0,0,  0,1,0,0,  0,0,0,1},
 {0,0,0,1,  0,1,0,0,  1,0,0,0,  0,0,1,0},
 {1,0,0,0,  0,0,1,0,  0,0,0,1,  0,1,0,0},
 {0,1,0,0,  0,0,0,1,  0,0,1,0,  1,0,0,0}};BN = Transpose[NB];
g11 = 20 g[1,1] + t[9] - 3 t[10] + 3 t[11] - t[12];
g12 = 20 g[1,2] + t[1] - 3 t[2] + 3 t[3] - t[4];
g13 = 20 g[1,3] + t[13] - 3 t[14] + 3 t[15] - t[16];
g14 = 20 g[1,4] + t[5] - 3 t[6] + 3 t[7] - t[8];
g21 = 20 g[2,1] + 3 t[2] - 3 t[5] - t[12] + t[15];
```

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g22 = 20 g[2,2] + t[4] - t[7] - 3 t[10] + 3 t[13];
g23 = 20 g[2,3] - t[1] + t[6] + 3 t[11] - 3 t[16];
g24 = 20 g[2,4] - 3 t[3] + 3 t[8] + t[9] - t[14];
g31 = 20 g[3,1] + 3 t[1] + t[8] - 3 t[10] - t[15];
g32 = 20 g[3,2] + 3 t[2] + t[7] - 3 t[9] - t[16];
g33 = 20 g[3,3] + 3 t[3] + t[6] - 3 t[12] - t[13];
g34 = 20 g[3,4] + 3 t[4] + t[5] - 3 t[11] - t[14];
g41 = 20 g[4,1] + 3 t[4] + t[8] - t[12] - 3 t[16];
g42 = 20 g[4,2] + t[3] + 3 t[7] - 3 t[11] - t[15];
g43 = 20 g[4,3] - 3 t[1] - t[5] + t[9] + 3 t[13];
g44 = 20 g[4,4] - t[2] - 3 t[6] + 3 t[10] + t[14];
g51 = 20 g[5,1] - 3 t[3] + t[5] + 3 t[10] - t[16];
g52 = 20 g[5,2] + t[4] - 3 t[6] - t[9] + 3 t[15];
g53 = 20 g[5,3] + 3 t[1] - t[7] - 3 t[12] + t[14];
g54 = 20 g[5,4] - t[2] + 3 t[8] + t[11] - 3 t[13];NG =
{{ 0, 0, 0, 0,  0, 0, 0, 0,  1,-3, 3,-1,  0, 0, 0, 0},
 { 1,-3, 3,-1,  0, 0, 0, 0,  0, 0, 0, 0,  0, 0, 0, 0},
 { 0, 0, 0, 0,  0, 0, 0, 0,  0, 0, 0, 0,  1,-3, 3,-1},
 { 0, 0, 0, 0,  1,-3, 3,-1,  0, 0, 0, 0,  0, 0, 0, 0},
 { 0, 3, 0, 0, -3, 0, 0, 0,  0, 0, 0,-1,  0, 0, 1, 0},
 { 0, 0, 0, 1,  0, 0,-1, 0,  0,-3, 0, 0,  3, 0, 0, 0},
 {-1, 0, 0, 0,  0, 1, 0, 0,  0, 0, 3, 0,  0, 0, 0,-3},
 { 0, 0,-3, 0,  0, 0, 0, 3,  1, 0, 0, 0,  0,-1, 0, 0},
 { 3, 0, 0, 0,  0, 0, 0, 1,  0,-3, 0, 0,  0, 0,-1, 0},
 { 0, 3, 0, 0,  0, 0, 1, 0, -3, 0, 0, 0,  0, 0, 0,-1},
 { 0, 0, 3, 0,  0, 1, 0, 0,  0, 0, 0,-3, -1, 0, 0, 0},
 { 0, 0, 0, 3,  1, 0, 0, 0,  0, 0,-3, 0,  0,-1, 0, 0},
 { 0, 0, 0, 3,  0, 0, 0, 1,  0, 0, 0,-1,  0, 0, 0,-3},
 { 0, 0, 1, 0,  0, 0, 3, 0,  0, 0,-3, 0,  0, 0,-1, 0},
 {-3, 0, 0, 0, -1, 0, 0, 0,  1, 0, 0, 0,  3, 0, 0, 0},
 { 0,-1, 0, 0,  0,-3, 0, 0,  0, 3, 0, 0,  0, 1, 0, 0},
 { 0, 0,-3, 0,  1, 0, 0, 0,  0, 3, 0, 0,  0, 0, 0,-1},
 { 0, 0, 0, 1,  0,-3, 0, 0, -1, 0, 0, 0,  0, 0, 3, 0},
 { 3, 0, 0, 0,  0, 0,-1, 0,  0, 0, 0,-3,  0, 1, 0, 0},
 { 0,-1, 0, 0,  0, 0, 0, 3,  0, 0, 1, 0, -3, 0, 0, 0}};
 GN = Transpose[NG];NG.GN;
 t1 = r t[1] + b[1,2] + b[2,3] + b[3,1] + b[4,3] + b[5,3] +
 g[1,2] - g[2,3] + 3 g[3,1] - 3 g[4,3] + 3 g[5,3] - S;
 t2 = r t[2] + b[1,2] + b[2,1] + b[3,2] + b[4,4] + b[5,4] -
 3 g[1,2] + 3 g[2,1] + 3 g[3,2] - g[4,4] - g[5,4] - S;
 t3 = r t[3] + b[1,2] + b[2,4] + b[3,3] + b[4,2] +  b[5,1] +
 3 g[1,2] - 3 g[2,4] + 3 g[3,3] + g[4,2] - 3 g[5,1] - S;
 t4 = r t[4] + b[1,2] + b[2,2] + b[3,4] + b[4,1] + b[5,2] -
 g[1,2] + g[2,2] + 3 g[3,4] + 3 g[4,1] + g[5,2] - S;
 t5 = r t[5] + b[1,4] + b[2,1] + b[3,4] + b[4,3] + b[5,1] +
 g[1,4] - 3 g[2,1] + g[3,4] - g[4,3] + g[5,1] - S;
 t6 = r t[6] + b[1,4] + b[2,3] + b[3,3] + b[4,4] + b[5,2] -
 3 g[1,4] + g[2,3] + g[3,3] - 3 g[4,4] - 3 g[5,2] - S;
 t7 = r t[7] + b[1,4] + b[2,2] + b[3,2] + b[4,2] + b[5,3] +
```

```
3 g[1,4] - g[2,2] + g[3,2] + 3 g[4,2] - g[5,3] - S;
t8 = r t[8] + b[1,4] + b[2,4] + b[3,1] + b[4,1] + b[5,4] -
g[1,4] + 3 g[2,4] + g[3,1] + g[4,1] + 3 g[5,4] - S;
t9 = r t[9] + b[1,1] + b[2,4] + b[3,2] + b[4,3] + b[5,2] +
g[1,1] + g[2,4] - 3 g[3,2] + g[4,3] - g[5,2] - S;
t10= r t[10]+ b[1,1] + b[2,2] + b[3,1] + b[4,4] + b[5,1] -
3 g[1,1] - 3 g[2,2] -3 g[3,1] + 3 g[4,4] + 3 g[5,1] - S;
t11= r t[11]+ b[1,1] + b[2,3] + b[3,4] + b[4,2] + b[5,4] +
3 g[1,1] + 3 g[2,3] -3 g[3,4] - 3 g[4,2] + g[5,4] - S;
t12= r t[12]+ b[1,1] + b[2,1] + b[3,3] + b[4,1] + b[5,3] -
g[1,1] - g[2,1] - 3 g[3,3] - g[4,1] - 3 g[5,3] - S;
t13= r t[13]+ b[1,3] + b[2,2] + b[3,3] + b[4,3] + b[5,4] +
g[1,3] + 3 g[2,2] - g[3,3] + 3 g[4,3] - 3 g[5,4] - S;
t14= r t[14]+ b[1,3] + b[2,4] + b[3,4] + b[4,4] + b[5,3] -
3 g[1,3] - g[2,4] - g[3,4] + g[4,4] + g[5,3] - S;
t15= r t[15]+ b[1,3] + b[2,1] + b[3,1] + b[4,2] + b[5,2] +
3 g[1,3] + g[2,1] - g[3,1] - g[4,2] + 3 g[5,2] - S;
t16= r t[16]+ b[1,3] + b[2,3] + b[3,2] + b[4,1] + b[5,1] -
g[1,3] - 3 g[2,3] - g[3,2] - 3 g[4,1] - g[5,1] - S;
B={{b11},{b12},{b13},{b14},{b21},{b22},{b23},{b24},{b31},{b32},
{b33},{b34},{b41},{b42},{b43},{b44},{b51},{b52},{b53},{b54}};
G={{g11},{g12},{g13},{g14},{g21},{g22},{g23},{g24},{g31},{g32},
{g33},{g34},{g41},{g42},{g43},{g44},{g51},{g52},{g53},{g54}};
T = {{t1},{t2},{t3},{t4},{t5},{t6},{t7},{t8},{t9},{t10},
{t11},{t12},{t13},{t14},{t15},{t16}};J0 =
{{0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0, 1,1,1,1},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0},
 {1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1, 0,0,0,0}};
Irk = IdentityMatrix[20];Iv = IdentityMatrix[16];
Jv = Table[1,{i,1,16},{j,1,16}];
A0B = Inverse[r Iv - GN.NG/20];
A = Inverse[r Iv - BN.NB/k + Jv/k];
```

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B0G = Inverse[20 Irk - NG.A.GN];
B0B = Inverse[k Irk - NB.A0B.BN + J0/r];
RHB = B - NB.A0B.(T - GN.G/20);
RHG = G - NG.A.(T - BN.B/k);
res = {g[f_,h_] g[f_,h_]->GR,g[f_,h_] g[i_,j_]->0,
       b[g_,h_] b[g_,h_]->BL,b[g_,h_] b[i_,j_]->0,
       b[g_,h_] g[g_,h_]->0, b[g_,h_] g[i_,j_]->0};
GTe = B0G.RHG;
BTe = B0B.RHB;
Expand[Transpose[GTe].RHG]/.res;
Expand[Transpose[BTe].RHB]/.res
```

$$\{\{\frac{413177907339875760\ BL}{9785630262982549}\}\}$$